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Functional analysis 2
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Sheet 5

Exercises for 'Functional Analysis 2' [MATH-404]

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Ex 5.0 (A non-trivial result from linear algebra [optional]) Let X be a vector space and $f_1, \dots, f_n, f : X \to \mathbb{R}$ be linear functionals. Show that the following properties are equivalent :

a) there exist $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that $f = \sum_{i=1}^n \lambda_i f_i$.

b) $\bigcap_{i=1}^n \mathrm{Ker}(f_i) \subset \mathrm{Ker}(f)$.

Hint: For the nontrivial implication consider the mapping $\Phi((f_1(x), \dots, f_n(x))) := f(x)$. Show that it is well-defined on $(f_1, \dots, f_n)(X) \subset \mathbb{R}^n$ and extend it.

Solution 5.0 : The implication a) $\implies$ b) is trivial. Therefore assume that b) holds true. Define $G : X \to \mathbb{R}^n$ by $G(x) = \sum_{i=1}^n f_i(x) e_i$. Then G is linear. Note that due to b) the condition $G(x_1) = G(x_2)$ implies that $f(x_1) = f(x_2)$. Hence the map $\Phi(G(x)) := f(x)$ is a well-defined linear map on $G(X) \subset \mathbb{R}^n$. Extend it to a linear map $\Phi' : \mathbb{R}^n \to \mathbb{R}$ (which requires no advanced result since $\mathbb{R}^n$ is finite-dimensional). Then for all $x \in X$,

$$f(x) = \Phi'(G(x)) = \sum_{i=1}^n f_i(x) \Phi'(e_i),$$

so the claim follows with the choice $\lambda_i = \Phi'(e_i)$.

Ex 5.1 (On the weak*-topology on a TVS)
Let X be a TVS with dual space X' and denote the weak*-topology on X' by τ' (cf. Definition 1.32).

a) Show that (X', τ') is a locally convex topological vector space.

b) Show that a sequence $(x'_n)_{n \in \mathbb{N}}$ converges to x' in (X', τ') if and only if $x'_n(x) \to x'(x)$ for all $x \in X$.

c)* Let X be a locally convex topological vector space. Show that (X', τ') is metrizable if and only if X has a countable algebraic base.
Hint: Recall Theorem 1.17 and use Exercise 5.0 for certain functionals on X'.

Solution 5.1 : a) Define the following seminorms on X' : for any $x \in X$ we set $p_x : X' \to [0, +\infty)$ as $p_x(x') = |x'(x)|$, which clearly is a seminorm. Note that when $p_x(x') = 0$ for all $x \in X$, then $x'(x) = 0$ for all $x \in X$, which implies $x' = 0$. Hence the topology τ_p induced by the seminorms $(p_x)_{x \in X}$ turns X' into a locally convex topological vector space. We show that $\tau' = \tau_p$, which implies the statement in a). Fix $U \in \tau_p$. Then for every $x' \in U$ there exist $x_1, \dots, x_n \in X$ such that

$$U \supset \bigcap_{i=1}^n \{p_{x_i}(y' - x') < \varepsilon\} = \bigcap_{i=1}^n \{|y'(x_i) - x'(x_i)| < \varepsilon\} = \bigcap_{i=1}^n \{y'(x_i) \in B_\varepsilon(x'(x_i))\}.$$

By definition of the weak*-topology the latter set is open in τ', hence $U \in \tau'$. To prove the converse inclusion, it is sufficient to show that linear maps $\phi_x : x' \mapsto x'(x)$ are continuous with respect to τ_p for every $x \in X$ (since by definition τ' is the coarsest topology with respect to which all these maps are continuous). It is sufficient to prove continuity at 0 so consider $B_\varepsilon(0) \subset \mathbb{R}$. We have

$$\phi_x^{-1}(B_\varepsilon(0)) = \{x' \in X' : |x'(x)| < \varepsilon\} = \{x' \in X' : p_x(x') < \varepsilon\},$$

which is an open set in τ_p. Thus $\tau' \subset \tau_p$ and we are done.

b) Let $(x'_n)_{n\in\mathbb{N}} \subset X'$ be a sequence such that $x'_n \to x'$ in the weak*-topology. Since continuity always implies sequential continuity, we get that $\phi_x(x'_n) \to \phi_x(x')$ for all $x \in X$, i.e. $x'_n(x) \to x'(x)$.

Conversely, assume that $x'_n(x) \to x'(x)$ for all $x \in X$. In particular, for any $x_1, \dots, x_k \in X$ it holds that $\max_{1\le i\le k} |x'_n(x_i) - x'(x_i)| \to 0$ as $n \to +\infty$. By the definition of the topology τ_p defined in a) and the fact that $\tau' = \tau_p$ we conclude that $x'_n \to x'$ in (X', τ').

c) Assume that τ' is metrizable. By Theorem 1.17 and the solution of a), there exists $(x_n)_{n\in\mathbb{N}} \subset X$ such that the seminorms $(p_{x_n})_{n\in\mathbb{N}}$ generate the topology τ'. Set $X_0 = \operatorname{span}(\{x_n : n \in \mathbb{N}\}) \subset X$. We claim that $X_0 = X$. Fix $x \in X$. Then the set $\{x' \in X' : |x'(x)| < 1\}$ contains a set of the form $\{x' \in X' : p_{x_i}(x') < \delta \text{ for all } i = 1, ..., k\}$. In other words,

$$\max_{1\le i\le k} |x'(x_i)| < \delta \implies |x'(x)| < 1$$

or by scaling,

$$\max_{1\le i\le k} |x'(x_i)| < \lambda\delta \implies |x'(x)| < \lambda.$$

We want to use Exercise 5.0. If $|x'(x_i)| = 0$ for all i, then the above implication implies that $|x'(x)| < \lambda$ for all $\lambda > 0$, so $x'(x) = 0$. Thus $\bigcap_{i=1}^k \operatorname{Ker}(\phi_{x_i}) \subset \operatorname{Ker}(\phi_x)$. From Exercise 5.0 we deduce that there exist $\lambda_1, \dots, \lambda_k \in \mathbb{K}$ such that

$$\phi_x = \sum_{i=1}^k \lambda_i \phi_{x_i}$$

i.e., for all $x' \in X'$ it holds

$$x'(x) = \sum_{i=1}^k \lambda_i x'(x_i) = x'\left(\sum_{i=1}^k \lambda_i x_i\right) \quad \implies \quad x'\left(x - \sum_{i=1}^k \lambda_i x_i\right) = 0$$

By Corollary 1.31, we deduce $x = \sum_{i=1}^k \lambda_i x_i$ since X is a LCTVS. Thus every element of X is a finite linear combination of elements of $\{x_i\}_{i\in\mathbb{N}}$, and we can inductively reduce this sequence to a linearly independent sequence by throwing out any x_k which are in the linear span of the preceeding $x_1, ..., x_{k-1}$ to obtain an algebraic basis.

To prove the reverse direction, let $(x_n)_{n\in\mathbb{N}}$ be an algebraic base for x and consider the seminorms $(p_{x_n})_{n\in\mathbb{N}}$. We show that these seminorms generate the weak*-topology. By a) it suffices to show that for every $x \in X$ there exist $x_1, \dots, x_k$ and a constant $c > 0$ such that $p_x \le c \max_{1\le i\le k} p_{x_i}$. To this end write $x = \sum_{i=1}^k \lambda_i x_i$ with $\lambda_i \in \mathbb{R}$ and $k \in \mathbb{N}$. Then

$$p_x(x') = |x'(x)| \le \sum_{i=1}^k \lambda_i |x'(x_i)| \le \underbrace{\left(\sum_{i=1}^k \lambda_i\right)}_{=:c} \max_{1\le i\le k} p_{x_i}(x').$$

Ex 5.2 (On continuity of differentiation and multiplication in $C^\infty(\Omega)$ and $\mathcal{D}_K$)
Let $\Omega \subset \mathbb{R}^d$ be open and $K \subset \mathbb{R}^d$ be compact. Show that

a) for any $\alpha \in \mathbb{N}_0^d$, the mapping $D^\alpha \colon \varphi \mapsto D^\alpha \varphi$ is a continuous linear operator in both $C^\infty(\Omega)$ and $\mathcal{D}_K$;
b) for any $f \in C^\infty(\mathbb{R}^d)$, the mapping $M_f \colon \varphi \mapsto f\varphi$ is a continuous linear operator in both $C^\infty(\Omega)$ and $\mathcal{D}_K$.

Solution 5.2 :
a) Clearly D^α and M_f are linear operators from $C^\infty(\Omega)$ to itself. Since, $\operatorname{supp}(D^\alpha\varphi) \subset \operatorname{supp}(\varphi)$ and $\operatorname{supp}(f\varphi) \subset \operatorname{supp}(\varphi)$, the same is true for $\mathcal{D}_K$.

As for the continuity in $C^\infty(\Omega)$, let us fix a family of appropriate compact sets $(K_n)_{n\in\mathbb{N}}$ and let $N \in \mathbb{N}$. We have

$$\begin{aligned} p_N(D^\alpha\varphi) &= \max\{|D^\beta D^\alpha \varphi(x)| : \ |\beta| \le N, x \in K_N\} \\ &\le \max\{|D^\gamma \varphi(x)| : \ |\gamma| \le N + |\alpha|, x \in K_{N+|\alpha|}\} = p_{N+|\alpha|}(\varphi). \end{aligned}$$

This shows the continuity of D^α. By the Leibniz rule, we have also

$$\begin{aligned} p_N(f\varphi) &= \max\{|D^\alpha(f\varphi)(x)| : \ |\alpha| \le N, x \in K_N\} \\ &\le \max\Big\{\sum_{\beta\le\alpha} |c_{\alpha,\beta} D^\beta f(x) D^{\alpha-\beta}\varphi(x)| : \ |\alpha| \le N, x \in K_N\Big\} \\ &\le C_N\, p_N(f|_\Omega)\, p_N(\varphi) \end{aligned}$$

for a suitably large constant C_N. This shows the continuity of M_f.

The corresponding result for $\mathcal{D}_K$ is immediate once we substitute K_N with K in the preceding estimates. Alternatively we can use the following general result from topology : *Let (X, τ) be a topological space, $Y \subset X$ and $f : X \to X$. If f is continuous with respect to τ and $f(Y) \subset Y$, then $f|_Y : Y \to Y$ is continuous with respect to the relative topology τ_Y.*

Ex 5.3 (An incomplete locally convex topology on test functions*)
Let $\Omega \subset \mathbb{R}^d$ be open and consider the set of test functions $\mathcal{D}(\Omega)$ equipped with the family of norms

$$\|\varphi\|_n = \max\{|D^\alpha\varphi(x)| : \ |\alpha| \le n, \ x \in \Omega\}, \quad n \in \mathbb{N}. \qquad (\star)$$

a) Assume that $\Omega = \mathbb{R}$. Pick $\varphi \in \mathcal{D}(\Omega)$ with $\operatorname{supp}(\varphi) = [0, 1]$ and $\varphi > 0$ on $(0, 1)$. Define

$$\psi_m(x) = \sum_{i=1}^{m} \frac{1}{i}\varphi(x - i).$$

Show that $(\psi_m)_m$ is a Cauchy sequence in $\mathcal{D}(\mathbb{R})$, but the pointwise limit $\psi_\infty = \lim \psi_m$ does not have compact support, hence it is not in $\mathcal{D}(\mathbb{R})$.

b) Show that for any open set $\Omega \subset \mathbb{R}^d$, the space $\mathcal{D}(\Omega)$ with the suggested topology is not complete.

Hint: For $\Omega \neq \mathbb{R}^d$, consider disjoint balls accumulating at the boundary and construct a sequence of functions appropriately modifying the example in item a).

Ex 5.4 (Heine–Borel and normability)
Recall that a TVS X has the *Heine-Borel property* if every bounded and closed subset of X is compact.

a) Prove that every normed vector space $(X, \|\cdot\|)$ that has the Heine–Borel property is finite dimensional.
   Hint: Recall Exercise 4.4.
b) Deduce that if $K \subset \mathbb{R}^d$ is compact with non-empty interior, the space $\mathcal{D}_K$ is not normable.

Solution 5.4 :
a) Let X be an infinite dimensional normed vector space. Since any normed space is Hausdorff, we can use Exercise 4.4 to see that no neighborhood of the origin in X is compact. Then $\overline{B_1(0)} := \{x \in X : \|x\| \leq 1\}$ is clearly closed and bounded, but as it is also a neighborhood of the origin it cannot be compact. Thus X does not have the Heine–Borel property.

b) Since $\mathcal{D}_K$ is infinite-dimensional (check that), to deduce that it is not normable it suffices to show that it has the Heine–Borel property. $C^\infty(\Omega)$ has the Heine–Borel property by Proposition 2.5 ; since $\mathcal{D}_K$ is a closed subset of $C^\infty(\Omega)$, it also inherits this property (for a direct proof, see Exercise 3.4 c)).